June 2, 2009

Econ 4240, ordinary exam, May 2009

Annotated solution ("sensorveiledning")

Problem 1

Many exams start with questions of this type. They should be fairly easy, in particular, because many of them have been given earlier, and the students have had the opportunity to look at the questions before.

(a) True.
A strategy which is part of a Nash equilibrium, cannot be strictly dominated. Hence it will never be eliminated during the process of iterated elimination of strictly dominated strategies, and it is rationalizable. If there were more than one Nash equilibrium, therefore, there must be more than one rationalizable strategy for at least one player.

(a) False. Counterexample:

	X	Y	Z
A	2,2	0,0	0,0
B	0,0	1,-1	-1,1
C	0,0	-1,1	1,-1

There is a unique Nash equilibrium, (A,X). No strategy is strictly dominated. Hence all strategies are rationalizable.

(c) False.
There always exists a subgame perfect Nash equilibrium, which can be found by backward induction. It need not be unique, however, since there can exist nodes at which the player is indifferent between two or more actions.

(d) False.
A Nash equilibrium can depend on a non-credible promise or threat and therefore not be subgame perfect.

Example involving a threat: Let player 1 choose between A and B. If A is chosen, the game ends with payoffs (1,4). If B is chosen, player 2 chooses between X and Y. B followed by X gives payoffs (0,2), B followed by Y gives payoffs (2,3), The normal form is:

	X	Y
A	1,4	1,4
B	0,2	2,3

Backward induction gives (B,Y). Both (A,X) and (B,Y) are Nash equilibria. X is not optimal in the subgame, and (A,X) is not subgame perfect. Implicitly, (A,X) amounts to player 2 making the following threat: "You better choose A; otherwise, I shall punish you by choosing X."

Problem 2

For given a and s_2, u_1 is a strictly concave function of s_1. Hence there is a unique value of s_1 that maximizes u_1; either an internal solution with $s_1 > 0$ and $\partial u_1 / \partial s_1 = 0$ or a corner solution with $s_1 = 0$ and $\partial u_1 / \partial s_1 \leq 0$. Since $\partial u_1 / \partial s_1 = 2 + 2as_2 - 2s_1$, the best response-function for player 1 is given by:

$$B_1(s_2) = \max\,(1 + as_2, 0)$$

Everything is symmetric, and the best response-function for player 2 is:

$$B_2(s_1) = \max\,(1 + as_1, 0)$$

An equilibrium (s_1^*, s_2^*) is characterized by $B_1(s_2^*) = s_1^*$ and $B_2(s_1^*) = s_2^*$.

We can draw the best-response functions in a (s_1, s_2)-diagram.

We see that the functions do not cross in the positive quadrant when $a \geq 1$. Hence there is no equilibrium in this case. From any point (s_1, s_2), at least one of the players can do better by a unilaterally change of strategy. In particular, if $s_1 \leq s_2$, player 1 can gain by increasing s_1; if $s_1 \geq s_2$, player 2 can gain by increasing s_2.

If $a < 1$, the functions cross in the positive quadrant. There is a symmetric equilibrium, given by

$$s_1^* = s_2^* = \frac{1}{1-a}$$

[Digression: For a = -1, $(s, 1-s)$ is an equilibrium for all $0 \leq s \leq 1$. This includes the symmetric equilibrium already mentioned. For a < -1, (0,1) and (1,0) are equilibria, in addition to the symmetric equilibrium.]

(a) The game has no equilibrium for $a \geq 1$.

(b) For $a = 0$, u_1 does not depend on s_2, and u_2 does not depend on s_1. At the equilibrium $s_1^* = s_2^* = 1$, therefore, u_1 and u_2 achieve their maximal value in the whole plane, and the equilibrium is efficient.

For $a \neq 0$, $a < 1$, the symmetric equilibrium found above is not efficient. For $0 < a < 1$, both players gain if s_1^* and s_2^* are increased by the same amount. For $a < 0$, both players gain if s_1^* and s_2^* are decreased by the some small and equal amount. (The students were not asked about this.)

The problems contain no reference to mixed strategies, and the students are not expected to bring up that issue. If player 2 chooses a mixed strategy with expected value s_2, the unique best response for player 1 is $B_1(s_2)$ as given above. Hence the best response is always unique, and no equilibrium can contain a mixed strategy.

Problem 3

The pure-strategy equilibria are of course (O,F) and (F,O). The mixed-strategy equilibrium is given by $p = \frac{5}{7}$, making player 2 indifferent between O and F, and $q = \frac{2}{7}$, making player 1 indifferent between O and F.

The mixed-strategy equilibrium is essentially symmetric, and in that sense it is more appealing than the pure-strategy equilibria, since the latter are asymmetric, and there is no basis for choosing one of them rather than the other. On the other hand, the mixed-strategy equilibrium relies on each player choosing a carefully chosen mix of O and F, in spite of being indifferent between them. Therefore, the concept of mixed-strategy equilibrium is rather tenuous.

Good students should mention both those considerations, which point in opposite directions.

Problem 4

As long as the principal and the agent cooperate, they receive a net payoff per period of $q-t-k$ and $t-c(q)$, respectively.

The use of trigger strategies implies that as soon as one of the parties has failed to cooperate in one period, there will be no cooperation thereafter. This applies even if the court enforces the contract. (It must be perfectly acceptable if a student makes a different assumption here.)

If the principal fails to cooperate by not paying *t*. there is nevertheless a probability *v* that the principal will be sentenced by the court to pay *t*. The one-period benefit from not cooperating, will therefore be $(1-v)t$, while the everlasting loss – staring one period later – is $q-t-k$. That is, deviating from cooperation is not advantageous if

$$(1-v)t \le \frac{\delta}{1-\delta}(q-t-k)$$

The corresponding inequality for the agent is

$$(1-v)c(q) \le \frac{\delta}{1-\delta}(t-c(q))$$

If both these inequalities are satisfied, cooperation is a subgame perfect equilibrium. The higher the value of *v*, that is, the better the court is able to enforce the contract, the better are the prospects for this being the case.

Problem 5

The problem follows quite closely the theory presented in Laffont & Martimort Chapter 4, Sections 1 – 3.

(a) The agent's expected utility of doing the job must be at least as high as the utility the agent can achieve by seeking alternative employment. If the latter is normalized to 0 and the transfer in case of a good and a bad outcome are denoted $\bar{t}$ and $\underline{t}$, respectively, the participation constraint for high effort is

$$\pi_1 u(\bar{t}) + (1-\pi_1) u(\underline{t}) \geq \psi$$

Cf. L&M eq. (4.4).

(b) The left-hand side is the expected added value of the agent exerting high effort rather than low. The right-hand side is the monetary compensation needed to make the agent indifferent between exerting high effort and receiving the compensation, and exerting low effort without the compensation. Hence this is the criterion for high effort being socially optimal.

(c) The transfer should be independent of the outcome. The agent will choose $e = 0$. The participation constraint is then $u(t) \geq 0$ (when the reservation utility is normalized to 0).

(d) High effort is socially optimal and can be induced by making $\bar{t} - \underline{t}$ sufficiently large. The level of $\bar{t}$ and $\underline{t}$ can be adjusted so that the participation constraint in (a) holds with equality. The result is socially optimal and the principal receives the whole surplus.

Since the agent is risk neutral, the utility function u is linear, and there is no loss of generality in assuming that it is the identity function. (An adjustment of u must be matched by an adjustment of ψ.) The transfers $\bar{t}$ and $\underline{t}$ must satisfy

$$(\bar{t} - \underline{t})(\pi_1 - \pi_0) \geq \psi ,$$

For example, $\bar{t}$ and $\underline{t}$ can be chosen so that the agent takes the whole risk of the outcome being good or bad. That is, the agent receives the value of the output minus a constant.

The problem is that $\underline{t}$ may be negative. It may be legally (institutionally, normatively) difficult to implement a negative transfer.

(e) The participation constraint of (a) must be satisfied, and $\bar{t} - \underline{t}$ must be sufficiently high to induce high effort. The latter condition is called the moral hazard incentive constraint and looks like this

$$\left(u(\bar{t}) - u(\underline{t})\right)(\pi_1 - \pi_0) \geq \psi .$$

Cf. L&M eq. (4.3).

It is optimal for the principal to let both these conditions hold with equality. If the participation constraint is satisfied with strict inequality, both $\bar{t}$

and $\underline{t}$ can be reduced, keeping $u(\overline{t})-u(\underline{t})$ unchanged. This is obviously advantageous for the principal. If the moral hazard incentive constraint is not satisfied with equality, unnecessary risk is imposed on the risk-averse agent. The risk-neutral principal can gain in expected value by "insuring" the agent against this unnecessary risk, by reducing $\overline{t}$ and increasing $\underline{t}$, keeping the left-hand side of the participation constraint unchanged.

The principal will not always offer such a contract. Even if the condition of (b) is satisfied, the principal's expected profit under the contract described here, may be smaller than the expected profit of the fixed-wage contract of (c).

(f) If high effort shall be induced, risk must necessarily be imposed on the risk-averse agent, and this is a social loss.

If high effort is not induced, a potential gain is lost, the expected value of which exceeds the cost to the agent of exerting high effort.